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Traces of High-Energy Processes in Strong Magnetic Fields

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High-energy processes in strong magnetic fields or in relativistic plasmas need special techniques in the evaluation of the traces that arise from the contribution of the electrons that mediate the interaction. To augment the standard procedure which is inadequate in such cases, a REDUCE based program is discussed and presented.

1. INTRODUCTION

Meaningful radiative processes in a plasma environment of high-energy particles under the influence of an intense magnetic field, where $\omega_c = eH/mc \gg \omega_p = (4\pi ne^2/m)^{1/2}$, take into account the surrounding particles' atmosphere as well as the consequences of instabilities where mode conversion is possible [1, 2]. Under such conditions, the electrons play a significant role in mediating the various interactions [3] and therefore summation over electron spin states is required to compensate for our ignorance of the exact spin states. In such a procedure, the evaluation of the trace cannot follow the standard procedure due to the fact that the square of such an electron wavefunction is irreducible and cannot be diagonalized. An alternative method, that enables the evaluation of the trace, rests upon the expansion of that irreducible matrix in the 16 independent Γ matrices that form a complete set.

In the setup of the calculation the wavefunction of the electron is obtained from the solution of the Dirac equation in a magnetic field [4] which is

$$\psi = \psi(\mathbf{r}) \exp[-imc^2\eta\epsilon t/\hbar], \tag{1}$$

$$\psi(\mathbf{r}) = u_n(\xi) \exp[-\tfrac{1}{2}\xi^2] \exp i[k_x x + k_z z],$$

$$\epsilon = [1 + \chi^2 + 2n\theta]^{1/2}; \qquad n = 0, 1,...; \qquad \chi = P_z/mc; \qquad \theta = H/H_c\,,$$

$$H_c = m^2c^3/e\hbar = 4.144 \times 10^{13}\,\mathrm{G},$$

$$u_n(\xi) = \begin{bmatrix} C_1\bar{H}_n(\xi) \\ C_2\bar{H}_{n-1}(\xi) \\ C_3\bar{H}_n(\xi) \\ C_4\bar{H}_{n-1}(\xi) \end{bmatrix},$$

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$$\xi = \alpha^{1/2}y + k_x\alpha^{-1/2}; \qquad \alpha = \theta\lambda_c^{-2}; \qquad \lambda_c = \hbar/mc,$$

$$C_1 = a\bar{A}, \qquad C_2 = sa\bar{B}, \qquad C_3 = \eta sb\bar{A}, \qquad C_4 = \eta b\bar{B},$$

$$a^2 = \tfrac{1}{2}(1 + \eta\epsilon^{-1}), \qquad b^2 = \tfrac{1}{2}(1 - \eta\epsilon^{-1}),$$

$$\bar{A}^2 = \tfrac{1}{2}[1 + s\chi(\chi^2 + 2n\theta)^{-1/2}], \qquad \bar{B}^2 = \tfrac{1}{2}[1 - s\chi(\chi^2 + 2n\theta)^{-1/2}],$$

$\eta = \pm 1$, $s = \pm 1$ indicating positive and negative energies and the sign of the projection of the momentum component along the spin. $\bar{H}_n$ are the Hermite polynomials normalized to 1 in $(-\infty, \infty)$,

$$\bar{H}_n = \alpha^{1/4}\pi^{-1/4}2^{-n/2}(n!)^{-1/2}H_n .$$

2. Taking a Trace

Taking the trace in the calculation of any scattering process pertinent to the present discussion involves something of the form

$$\mathrm{Tr}\ \bar{\psi}A\psi\bar{\psi}B\psi$$

where A (or B) can have from one to five different Lorentz four vectors. For example, in the longitudinal to transverse mode conversion [2] (to be referred to as paper I) we find in one case that $A = \not{e}\not{k}\not{e}'$; $B = \not{e}'\not{k}\not{e}$ where e, e' are the appropriate polarization vectors of the plasma modes (longitudinal or transverse plasma photon), k comes from the propagator contribution to the momentum, and $\not{e} = \gamma^\mu e_\mu$. Or in the calculations of the emission vertex for a plasma mode and an electron (cf. paper I) $A = B = \not{e}$.

Further

$$\begin{aligned}\mathrm{Tr}\ \bar{\psi}A\psi\bar{\psi}B\psi &= \mathrm{const}\ \mathrm{Tr}\ \bar{U}AU\bar{U}BU \\ &= \mathrm{const}\ \mathrm{Tr}\ AMBM'\end{aligned}$$

where M is a 4×4 matrix, the origin of which can be easily traced to the form of the electron wavefunction satisfying the Dirac equation in a strong magnetic field. Namely,

$$U_f\bar{U}_f = \begin{bmatrix} c_1{}^2\bar{H}_n{}^2 & 0 & c_1c_3\bar{H}_n{}^2 & 0 \\ 0 & c_2\bar{H}^2_{n-1} & 0 & c_2c_4\bar{H}^2_{n-1} \\ c_3c_1\bar{H}_n{}^2 & 0 & c_3{}^2\bar{H}_n{}^2 & 0 \\ 0 & c_4c_2\bar{H}^2_{n-1} & 0 & c_4{}^2\bar{H}^2_{n-1} \end{bmatrix}, \tag{2a}$$

$$U_i\bar{U}_i = \begin{bmatrix} c_2'^2\bar{H}^2_{n'} & 0 & c_1'c_3'\bar{H}^2_{n'} & 0 \\ 0 & c_2'^2\bar{H}^2_{n'-1} & 0 & c_2'c_4'\bar{H}^2_{n'-1} \\ c_3'c_1'\bar{H}^2_{n'} & 0 & c_3'^2\bar{H}^2_{n'} & 0 \\ 0 & c_4'c_2'\bar{H}^2_{n'-1} & 0 & c_4'^2\bar{H}^2_{n'-1} \end{bmatrix}. \tag{2b}$$

It is essentially the information contained in M, information about transitions among Landau levels and radiation, and its particular form which makes the processes in a strong magnetic field different from the standard calculations. To proceed with the calculations $U\bar{U} = M$ is expanded in the 16 different Γ matrices which constitute a complete set

$$M = aI + \gamma^\mu b_\mu + \gamma_5\gamma^\mu c_\mu + d\gamma_5 + e_{\mu\nu}\sigma^{\mu\nu} \tag{3}$$

with the coefficients a, b_μ, c_μ, d, $e_{\mu\nu}$ and where

$$\sigma^{\mu\nu} = (i/2)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu).$$

To single out the proper coefficients we multiply M by the appropriate matrix and take a trace to find that

$$a = \tfrac{1}{4}\,\mathrm{Tr}\, M, \tag{4a}$$

$$b_\mu = \tfrac{1}{4}\,\mathrm{Tr}(\gamma_\nu M), \tag{4b}$$

$$c_\mu = -\tfrac{1}{4}\,\mathrm{Tr}(\gamma_5\gamma_\nu M), \tag{4c}$$

$$d = \tfrac{1}{4}\,\mathrm{Tr}(\gamma_5 M), \tag{4d}$$

$$e_{\rho\lambda} = \tfrac{1}{4}\,\mathrm{Tr}(\sigma_{\rho\lambda} M). \tag{4e}$$

It should be noted that while A and B change according to the particular case, there is always the same part of the combination M, M' that survives as the coefficients g_i ($i = 1,\ldots, 13$) that will further depend in each case on the combination of the Dirac delta functions that result from the integration.

For any one particular combination of A and B we have therefore to calculate the following terms:

$$g_1 AB, \quad g_2 AB\gamma_5, \quad g_3 AB\sigma^{\mu\nu}, \quad g_4 A\gamma_\mu B\gamma_\nu, \quad g_5 A\gamma_\mu B\gamma_5\gamma_\nu,$$
$$g_6 A\gamma_5\gamma_\mu B\gamma_\nu, \quad g_7 A\gamma_5\gamma_\mu B\gamma_5\gamma_\nu, \quad g_8 A\gamma_5 B, \quad g_9 A\gamma_5 B\gamma_5,$$
$$g_{10} A\gamma_5 B\sigma^{\mu\nu}, \quad g_{11} A\sigma^{\mu\nu} B, \quad g_{12} A\sigma^{\mu\nu} B\gamma_5, \quad g_{13} A\sigma^{\mu\nu} B\sigma^{\rho\lambda}.$$

Implicit in some of the above there still is a sum over the Lorentz four indices which in fact is needed in order that the array of numbers obtained as the form of g_i be collapsed into a sum. Such a collapse will take place when summation over the paired indices is performed (an index in g_i and an index in the rest of the term). For example, in the case of g_4 we have

$$\begin{aligned}\mathrm{Tr}\, g_4 A\gamma_\mu B\gamma_\nu &= \mathrm{Tr}[\tfrac{1}{8}(\mathrm{Tr}\,\gamma_\mu M)(\mathrm{Tr}\,\gamma_\nu M')\, A\gamma_\mu B\gamma_\nu] \\ &= \tfrac{1}{8}\,\mathrm{Tr}\left(\sum_\mu \sum_\nu (\mathrm{Tr}\,\gamma_\mu M)(\mathrm{Tr}\,\gamma_\nu M')\, A\gamma_\mu B\gamma_\nu\right).\end{aligned} \tag{5}$$

3. Some Radiative Processes

More specifically, when the mode conversion of plasmon to photon is calculated, in essence what has to be performed is the calculation of four expressions that are of the form similar to [2]

$$\mathrm{Tr}\, \not{\epsilon}(i\not{D} + m)\, \not{\epsilon}' M \not{\epsilon}'(i\not{D} + m)\, \not{\epsilon} M' \tag{6a}$$

where

$$D = [(mc^2\epsilon_\beta - 2\hbar\omega')^2 - (P_x' - 2k_1')^2 - (P_z' - 2k_3')^2]^{1/2}.$$

The shortest nontrivial trace to be evaluated is therefore of the form (see Appendix A for the expanded form) Tr $\not{e}\not{e}'M\not{e}'\not{e}M'$ whereas we also have higher terms such as

$$\mathrm{Tr}\, \not{e}\not{D}\not{e}'(\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\mu\nu}M)\,\sigma^{\mu\nu}\not{e}'\not{D}\not{e}(\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\lambda\rho}M')\,\sigma^{\lambda\rho} \tag{6b}$$

which gives

$$-\tfrac{1}{2}\,\mathrm{Tr}(\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\mu\nu}M)(\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\lambda\rho}M')[\not{e}\not{D}\not{e}'\gamma^\mu\gamma^\nu\not{e}'\not{D}\not{e}\gamma^\lambda\gamma^\rho$$
$$-\not{e}\not{D}\not{e}'\gamma^\mu\gamma^\nu e'De\gamma^\rho\gamma^\lambda - \not{e}\not{D}\not{e}'\gamma^\nu\gamma^\mu\not{e}'\not{D}\not{e}\gamma^\lambda\gamma^\rho$$
$$+ \not{e}\not{D}\not{e}'\gamma^\nu\gamma^\mu\not{e}'\not{D}\not{e}\gamma^\rho\gamma^\lambda].$$

We see that a trace has to be performed over 10 Dirac matrices. In another case, however, we have

$$\mathrm{Tr}\, \not{e}\not{D}\not{e}'\gamma_5\gamma^\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\mu M)\,\not{e}'\not{D}\not{e}\gamma_5\gamma^\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\lambda M') \tag{7}$$

which involves 16 Dirac matrices once γ_5 is written in its explicit form of $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

If we consider a different process, say that of magneto-Bremssrhalung (cf. paper I) we find the expression

$$\mathrm{Tr}\, \not{e}(\not{D}' - m)\,\gamma_4 M\gamma_4(\not{D} - m)\,\not{e}M' \tag{8}$$

which gives similar-type terms.

There are two types of expressions in any such calculation. That involving an even number of contracted Dirac matrices, and the one involving an odd number of such matrices (not to be confused with the total number of all the Dirac matrices—contracted and noncontracted that are to be traced—as obviously in this case the trace of an odd number vanishes). The coefficients for those traces are defined as

$$g_1^{\text{even}},\dots, g_{13}^{\text{even}} \equiv g_1,\dots, g_{13} \qquad \text{and} \qquad g_1^{\text{odd}},\dots, g_{13}^{\text{odd}}$$

(see Appendix B). Examining the 25 terms in any (e.g., of the above) group, it is noted that apart from $g_2^{\text{even}} = g_3^{\text{odd}}$ the g_i^{even} and g_i^{odd} are complimentary.

The same situation prevails of course in other calculated processes such as in the 3 mode interaction under similar conditions, and thus is of a general nature which deserves a general computer programming that facilitates computational efforts of such high-energy processes.

Thus in the magneto-Bremssrhalung process we have that (8) is now

$$\begin{aligned}\mathrm{Tr}[(&g_1\not{e}\not{D}\gamma_4\gamma_4\not{D}\not{e} + g_2\not{e}\not{D}\gamma_4\gamma_4\not{D}\not{e}\gamma_5 + g_3\not{e}\not{D}\not{D}\not{e}\sigma^{\lambda\rho}\\ &+ g_4\not{e}\not{D}\gamma_4\gamma_\mu\gamma_4\not{D}\not{e}\gamma_\lambda + g_5\not{e}\not{D}\gamma_4\gamma_\mu\gamma_4\not{D}\not{e}\gamma_5\gamma^\lambda\\ &+ g_6\not{e}\not{D}\gamma_4\gamma_5\gamma^\mu\gamma_4\not{D}\not{e}\gamma_\lambda + g_7\not{e}\not{D}\gamma_4\gamma_5\gamma^\mu\gamma_4\not{D}\not{e}\gamma_5\gamma^\lambda\\ &+ g_8\not{e}\not{D}\gamma_4\gamma_5\gamma_4\not{D}\not{e}\gamma_5 + g_9\not{e}\not{D}\gamma_4\gamma_5\gamma_4\not{D}\not{e}\gamma_5\\ &+ g_{10}\not{e}\not{D}\gamma_4\gamma_5\gamma_4\not{D}\not{e}\sigma^{\lambda\rho} + g_{11}\not{e}\not{D}\gamma_4\sigma^{\mu\nu}\not{D}\not{e}\\ &+ g_{12}\not{e}\not{D}\gamma_4\sigma^{\mu\nu}\gamma_4\not{D}\not{e}\gamma_5 + g_{13}\not{e}\not{D}\gamma_4\sigma^{\mu\nu}\gamma_4\not{D}\not{e}\sigma^{\lambda\rho})\\ &+ 3 \text{ other expressions of the same nature}].\end{aligned} \tag{9}$$

When higher order processes are calculated such as the 3 plasmon interaction (cf. paper I) the situation is even more combersome as we now have

$$\mathrm{Tr}[\not{e}(i\not{D} + m)\not{e}'(i\not{D}' + m)\not{e}''M\not{e}''(i\not{D}' + m)\not{e}'(i\not{D} + m)\not{e}M' \\ + 5 \text{ more terms of the same nature}] \tag{10}$$

where

$$D' = [(mc^2\epsilon_\beta)^2 - P_x'^2 - P_z'^2]^{1/2}.$$

When the expansion for M and M' is employed in each of the resulting terms, the number of the γ matrices to be traced in one product goes as high as 20.

From the outset we would then have to evaluate for the 3 mode interaction 13 times, 16 terms in each of the total 6 expressions, which gives us 1248 terms where each single trace is by itself; not an easy task to be carried out.

4. Automated Computations

Sensibly such a calculation requires the aid of a computer. This, however, presents a computational problem even if computers are used, due to their present capabilities. The size of the core and output needed to evaluate traces does not grow linearly with the length of the term traced over, and a machine like the PDP 10, for example, will run into difficulties when traces over 10γ matrices are performed, and special software and hardware knowledge is required, apart from the construction of the program, to obtain a trace of, say, 12 Dirac matrices, let alone such a high number as 16 or 20.

In discussing the programming it is instructive and important first to point out the existing methods and then where and why they are inappropriate for our case. A less complicated case is of course the standard case which will take several months to be calculated manually [6] despite the simple algebraic nature of the calculation, with of course the numerous opportunities that exist for errors. In such standard calculations, automated computation is utilized. In fact the need for automated computation was first realized in the calculation of traces of products of Dirac matrices and four momenta in quantum electrodynamics. A simple high computational language such as FORTRAN is not flexible enough for such calculations, though interesting attempts to use FORTRAN for that purpose have been made with some success [7], and using the machine code for this purpose will result in large volumes which are quite tedious [6].

As a result there exist at present three different symbol manipulation systems that can normally be used when Dirac algebra is involved. ASHMEDAI, written by Levine [8]; SCHOONSCHIP, written by Veltman [9]; and REDUCE, written by Hearn [5].

Both ASHMEDAI and SCHOONSHIP are written in machine language. ASHMEDAI is written partly in the CDC 3000 machine language and partly in FORTRAN, and SCHOONSCHIP is written in the CDC 6000, 7000 machine code. REDUCE is based on Lisp 1.5 [6] rather than a machine code. However, due to the

Lisp 1.5 structure it is obvious that both run time and core would be significantly larger in the latter case than, say when using SCHOONSCHIP which affords the user the capability of handling large expressions while using less than 30k of core memory [10].

REDUCE in its standard form has successfully been used by authors such as Clark and his collaborators. For examples, see [11–13].

The present work is based on REDUCE. It differs from the standard procedure in that among other calculations, the inner trace has also to be performed in order to evaluate the coefficients in the expansion of $u\bar{u}$ (see Appendix B). Such coefficients cannot be calculated in a separate self-supporting manner, as the results yield terms that have the appropriate indices which ensure the possibility and allow the evaluation of the outer traces, so to speak. These coefficients contain in the form of $u\bar{u}$ numerical parts obtainable in a numeric calculation of the C_i terms (cf. (1) and (5)). Therefore it is a mixture of numeric and symbol manipulating languages that have to be used simultaneously. While in the past such a mixture could be obtained, for example, by using the machine language, which required more specific knowledge of the machine used (i.e., its internal software construction), for the calculation, as such a program was not machine independent, it is now readily accessible through the system, as it enables one to call in separate procedures written in either ALGOL or FORTRAN. The present program, though based on REDUCE, nevertheless requires new structure, layout, and definitions right from the outset.

In order to use the symbol manipulating program with the aid of REDUCE the reader is referred to Appendix C.

5. Alternatives

It should be noted that other ways exist by which a result can be obtained, though less elegant, less concise, and of course less general. For example, if the term Tr $\not{e}\not{D}\not{e}'M\not{e}'\not{D}\not{e}M'$ is to be so calculated, one obvious way of doing it is to require the explicit use of the conditions imposed by the $\delta_{n,n'}$ and $\delta_{n-1,n'}$ which are obtained from the y integration over the matrices M and M' that contain the appropriate Hermite functions. Thus it is found that

$$\Xi' = \not{D}\not{e}'M = \bar{\bar{\sigma}}$$

where $\bar{\bar{\sigma}}$ is the matrix of elements σ_{ij} ($i = 1,..., 4; j = 1,..., 4$) (cf. Appendix D for the explicit form of the different σ_{ij}). Obviously $\Xi_1 = \not{e}\Xi'$ is then obtained.

In the same way we find

$$\Xi'' = \not{D}\not{e}M' = \bar{\bar{\sigma}}'$$

where $\bar{\bar{\sigma}}'$ is the matrix of elements σ'_{ij} ($i = 1,..., 4; j = 1,..., 4$), and then move to obtain

$$\Xi_2 = \not{e}'\Xi''. \tag{13}$$

Finally we take the product $\Xi = \Xi_1 \Xi_2$ and then the trace of it as a simple sum of the diagonal terms.

Results for the cases $\delta_{n',n}$ and $\delta_{n-1,n'}$ have to be obtained separately (cf. Appendix D). Only after that can order of magnitude calculations give final numerical answers to the problem under study.

Appendix A

The expanded form of the shortest nontrivial trace is:

$$\begin{aligned}
\mathrm{Tr}\{ & m^2 \not{e}\not{e}'[\tfrac{1}{4}\,\mathrm{Tr}\,M\mathbf{I} + \gamma_\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_\mu M) - \gamma_5\gamma^\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\mu M) + (\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M)\,\gamma_5 \\
& + (\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\mu\nu}M)\,\sigma^{\mu\nu}]\,\not{e}'\not{e}[\tfrac{1}{4}\,\mathrm{Tr}\,M'\mathbf{I} + \gamma_\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_\lambda M') - \gamma_5\gamma^\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\lambda M') \\
& + (\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M')\,\gamma_5 + (\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\lambda\rho}M')\,\sigma^{\lambda\rho}] + im(\not{e}\not{e}'[\tfrac{1}{4}\,\mathrm{Tr}\,M\mathbf{I} + \gamma_\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_\mu M) \\
& - \gamma_5\gamma^\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\mu M) + \tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M)\,\gamma_5 + (\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\mu\nu}M)\,\sigma^{\mu\nu}]\,\not{e}'\not{D}'\not{e}[\tfrac{1}{4}\,\mathrm{Tr}\,M'\mathbf{I} \\
& + \gamma_\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_\lambda M') - \gamma_5\gamma^\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\lambda M') + (\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M')\,\gamma_5 + (\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\lambda\rho}M')\,\sigma^{\lambda\rho}] \\
& + \not{e}\not{D}\not{e}'[\tfrac{1}{4}\,\mathrm{Tr}\,M\mathbf{I} + \gamma_\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_\mu M) - \gamma_5\gamma^\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\mu M) + (\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M)\,\gamma_5 \\
& + (\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\mu\nu}M)\,\sigma^{\mu\nu}]\,\not{e}'\not{e}[\tfrac{1}{4}\,\mathrm{Tr}\,M'\mathbf{I} + \gamma_\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_\lambda M') - \gamma_5\gamma^\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\lambda M') \\
& + (\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M')\,\gamma_5 + (\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\lambda\rho}M')\,\sigma^{\lambda\rho}]) - \not{e}\not{D}\not{e}'[\tfrac{1}{4}\,\mathrm{Tr}\,M\mathbf{I} + \gamma_\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_\mu M) \\
& - \gamma_5\gamma^\mu(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\mu M) + (\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M)\,\gamma_5 + (\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\mu\nu}M)\,\sigma^{\mu\nu}]\,\not{e}'\not{D}\not{e}[\tfrac{1}{4}\,\mathrm{Tr}\,M'\mathbf{I} \\
& + \gamma^\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_\lambda M') - \gamma_5\gamma^\lambda(\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5\gamma_\lambda M') + (\tfrac{1}{4}\,\mathrm{Tr}\,\gamma_5 M')\,\gamma_5 + (\tfrac{1}{4}\,\mathrm{Tr}\,\sigma_{\lambda\rho}M')\,\sigma^{\lambda\rho}]\}.
\end{aligned}$$

Appendix B

$$\begin{aligned}
& g_1 = \tfrac{1}{8}(\mathrm{Tr}\,M\mathbf{I})(\mathrm{Tr}\,M'\mathbf{I}), \qquad g_2 = \tfrac{1}{8}(\mathrm{Tr}\,M\mathbf{I})(\mathrm{Tr}\,\gamma_5 M'), \qquad g_3 = \tfrac{1}{8}(\mathrm{Tr}\,M\mathbf{I})(\mathrm{Tr}\,\sigma^{\mu\nu}M'); \\
& g_4 = \tfrac{1}{8}(\mathrm{Tr}\,\gamma_\mu M)(\mathrm{Tr}\,\gamma_\nu M'), \qquad g_5 = -\tfrac{1}{8}(\mathrm{Tr}\,\gamma_\mu M)(\mathrm{Tr}\,\gamma_5\gamma_\nu M'); \\
& g_6 = -\tfrac{1}{8}(\mathrm{Tr}\,\gamma_5\gamma_\mu M)(\mathrm{Tr}\,\gamma_\nu M'), \qquad g_7 = \tfrac{1}{8}(\mathrm{Tr}\,\gamma_5\gamma_\mu M)(\mathrm{Tr}\,\gamma_5\gamma_\nu M'); \\
& g_8 = \tfrac{1}{8}(\mathrm{Tr}\,\gamma_5 M)(\mathrm{Tr}\,M'\mathbf{I}), \qquad g_9 = \tfrac{1}{8}(\mathrm{Tr}\,\gamma_5 M)(\mathrm{Tr}\,\gamma_5 M'), \qquad g_{10} = \tfrac{1}{8}(\mathrm{Tr}\,\gamma_5 M)(\mathrm{Tr}\,\sigma^{\mu\nu}M'); \\
& g_{11} = \tfrac{1}{8}(\mathrm{Tr}\,\sigma^{\mu\nu}M)(\mathrm{Tr}\,M'\mathbf{I}), \qquad g_{12} = \tfrac{1}{8}(\mathrm{Tr}\,\sigma^{\mu\nu}M)(\mathrm{Tr}\,\gamma_5 M'), \\
& g_{13} = \tfrac{1}{8}(\mathrm{Tr}\,\sigma^{\mu\nu}M)(\mathrm{Tr}\,\sigma^{\rho\lambda}M').
\end{aligned}$$

In particular in our case

$$\begin{aligned}
& g_1 = \tfrac{1}{8}(\bar{C}_1^{\,2} + \bar{C}_2^{\,2} + \bar{C}_3^{\,2} + \bar{C}_4^{\,2})(\bar{C}_1'^2 + \bar{C}_2'^2 + \bar{C}_3'^2 + \bar{C}_4'^2); \\
& g_2 = \tfrac{1}{8}(\bar{C}_1^{\,2} + \bar{C}_2^{\,2} + \bar{C}_3^{\,2} + \bar{C}_4^{\,2})(\bar{C}_3'\bar{C}_1' + \bar{C}_4'\bar{C}_2' + \bar{C}_1'\bar{C}_3' + \bar{C}_2'\bar{C}_4'); \\
& g_3 = \tfrac{1}{8}(\bar{C}_1^{\,2} + \bar{C}_2^{\,2} + \bar{C}_3^{\,2} + \bar{C}_4^{\,2})\,X'_{\mu\nu}, \qquad g_4 = \tfrac{1}{8}X_\mu X'_\nu, \qquad g_5 = -\tfrac{1}{8}X_\mu Y'_\nu; \\
& g_6 = -\tfrac{1}{8}Y_\mu X'_\nu, \qquad g_7 = \tfrac{1}{8}Y_\mu Y'_\nu; \\
& g_8 = \tfrac{1}{8}(\bar{C}_3\bar{C}_1 + \bar{C}_4\bar{C}_2 + \bar{C}_1\bar{C}_3 + \bar{C}_2\bar{C}_4)(\bar{C}_1'^2 + \bar{C}_2'^2 + \bar{C}_3'^2 + \bar{C}_4'^2);
\end{aligned}$$

$$g_9 = \tfrac{1}{8}(\bar{C}_3\bar{C}_1 + \bar{C}_4\bar{C}_2 + \bar{C}_1\bar{C}_3 + \bar{C}_2\bar{C}_4)(\bar{C}'_3\bar{C}'_1 + \bar{C}'_4\bar{C}'_2 + \bar{C}'_1\bar{C}'_3 + \bar{C}'_2\bar{C}'_4);$$

$$g_{10} = \tfrac{1}{8}(\bar{C}_3\bar{C}_1 + \bar{C}_4\bar{C}_2 + \bar{C}_1\bar{C}_3 + \bar{C}_2\bar{C}_4)\, X'_{\mu\nu}\,;$$

$$g_{11} = \tfrac{1}{8}X_{\mu\nu}(\bar{C}_1'^2 + \bar{C}_2'^2 + \bar{C}_3'^2 + \bar{C}_4'^2);$$

$$g_{12} = \tfrac{1}{8}X_{\mu\nu}(\bar{C}'_3\bar{C}'_1 + \bar{C}'_4\bar{C}'_2 + \bar{C}'_1\bar{C}'_3 + \bar{C}'_2\bar{C}'_4);$$

$$g_{13} = \tfrac{1}{8}X_{\mu\nu}X'_{\rho\lambda}$$

where

$$x_\mu = (\bar{C}_1^2 + \bar{C}_2^2 - \bar{C}_3^2 - \bar{C}_4^2, 0, 0, \bar{C}_3\bar{C}_1 + \bar{C}_2\bar{C}_4 - \bar{C}_1\bar{C}_3 - \bar{C}_4\bar{C}_2);$$

$$y_\mu = (-\bar{C}_3\bar{C}_1 - \bar{C}_4\bar{C}_2 + \bar{C}_1\bar{C}_3 + \bar{C}_2\bar{C}_4\,, 0, 0, -\bar{C}_1^2 + \bar{C}_2^2 + \bar{C}_3^2 - \bar{C}_4^2);$$

$$\begin{aligned} x_{\mu\nu} = (&0, 0, 0, i[\bar{C}_3\bar{C}_1 - \bar{C}_4\bar{C}_2 + \bar{C}_1\bar{C}_3 - \bar{C}_2\bar{C}_4], 0, 0, [\bar{C}_1^2 + \bar{C}_3^2 - \bar{C}_2^2 - \bar{C}_4^2], 0,\\ &0, [-\bar{C}_1^2 - \bar{C}_3^2 + \bar{C}_2^2 - \bar{C}_4^2], 0, 0, i[-\bar{C}_3\bar{C}_1 + \bar{C}_4\bar{C}_2 - \bar{C}_1\bar{C}_3 + \bar{C}_2\bar{C}_4], 0, 0, 0)\end{aligned}$$

and where

$$\bar{C}_{\text{even}} = C_{\text{even}}\bar{H}_{n-1}\,; \qquad \bar{C}_{\text{odd}} = C_{\text{odd}}\bar{H}_n\,;$$

$$\bar{C}'_{\text{even}} = C'_{\text{even}}\bar{H}_{n'-1}\,; \qquad \bar{C}'_{\text{odd}} = C_{\text{odd}}\bar{H}_{n'}\,.$$

Appendix C

The following statements are given in the form of LET statements:

$$\begin{aligned} C11 &= C1 ** 2, & C11P &= C1P ** 2,\\ C22 &= C2 ** 2, & C22P &= C2P ** 2,\\ C33 &= C3 ** 2, & C33P &= C3P ** 2,\\ C44 &= C4 ** 2, & C44P &= C4P ** 2;\end{aligned}$$

$$\begin{aligned} C1C &= C1 ** 2 * C1P ** 2,\\ C2C &= C2 ** 2 * C2P ** 2,\\ C3C &= C3 ** 2 * C3P ** 2,\\ C4C &= C4 ** 2 * C4P ** 2;\end{aligned}$$

$$\begin{aligned} C1PC &= C1P ** C1 ** 2, & C11C &= C1C + C1PC,\\ C2PC &= C2P ** C2 ** 2, & C22C &= C2C + C2PC,\\ C3PC &= C3P ** C3 ** 2, & C33C &= C3C + C3PC,\\ C4PC &= C4P ** C4 ** 2, & C44C &= C4C + C4PC;\end{aligned}$$

$$C1234C = C1C + C2C + C3C + C4C;$$

$$\begin{aligned} C12C &= C1 ** 2 * C2P ** 2, & C21C &= C2 ** 2 * C1P ** 2,\\ C13C &= C1 ** 2 * C3P ** 2, & C13C &= C1 ** 2 * C3P ** 2,\\ C14C &= C1 ** 2 * C4P ** 2, & C41C &= C4 ** 2 * C1P ** 2,\\ C23C &= C2 ** 2 * C3P ** 2, & C32C &= C3 ** 2 * C2P ** 2,\\ C24C &= C2 ** 2 * C4P ** 2, & C42C &= C4 ** 2 * C2P ** 2,\\ C34C &= C3 ** 2 * C4P ** 2, & C43C &= C4 ** 2 * C3P ** 2,\end{aligned}$$

$$\begin{aligned} C1221C &= C12C + C21C,\\ C1331C &= C13C + C31C,\\ C1441C &= C14C + C41C,\\ C2332C &= C23C + C32C,\\ C2442C &= C24C + C42C,\\ C3443C &= C34C + C43C; \end{aligned}$$

$$\begin{aligned} G1 = \tfrac{1}{8} * (&C11 * C11P + C22 * C22P + C33 * C33P + C44 * C44P\\ &+ C11 * C22P + C22 * C11P + C11 * C33P + C33 * C11P\\ &+ C11 * C44P + C44 + C11P + C22 * C33P + C33 * C22P\\ &+ C22 * C44P + C44 * C22P + C33 * C44P + C44 * C33P),\\ G2 = \tfrac{1}{4} * (&C11 * C3P * C1P + C11 * C4P * C2P + C22 * C3P * C1P\\ &+ C22 * C4P * C2P + C33 * C3P * C1P + C33 * C4P * C2P\\ &+ C44 * C3P * C1P + C44 * C4P * C2P),\\ G8 = \tfrac{1}{4} * (&C11P * C3 * C1 + C11P * C4 * C2 + C22P * C3 * C1\\ &+ C22P * C4 * C2 + C33P * C3 * C1 + C33P * C4 * C2\\ &+ C44P * C3 * C1 + C44P * C4 * C2),\\ G9 = \tfrac{1}{2} * (&C3 * C1 * C3P * C1P + C3 * C1 * C4P * C2P\\ &+ C4 * C2 * C4P * C2P + C2 * C4 * C3P * C1P). \end{aligned}$$

$G3,\ldots, G7$ and $G10,\ldots, G13$ (inclusive) require the introduction of RULE statements in addition to the above LET statements (cf. Appendix B). Namely, in addition we have

RULE SIGMA $(L, MU, NU,) = \frac{1}{2} * (G(L, MU) * G(L, NU) - G(L, NU) * G(L, MU))$.

For $X_{\mu\nu}$ and $X'_{\nu\mu}$ we have

RULE $ZZ(L, MU, NU) = (Z.\ MU) * (Z.\ NU) - (Z.\ NU) * (Z.\ MU)$;

also

RULE $ZZP(L, MU, NU) = (ZP.\ MU) * (ZP.\ NU) - (ZP.\ NU) * (ZP.\ MU)$;

$$\begin{aligned} &Z(L, 0, 0) = 0 \quad Z(L, 0, 1) = 0, \quad Z(L, 0, 2) = 0,\\ &Z(L, 0, 3) = I * (C3 * C1 - C4 * C2 + C1 * C3 - C2 * C4),\\ &Z(L, 1, 0) = 0, \quad Z(L, 1, 1) = 0, \quad Z(L, 1, 2) = (C11 + C33 - C22 - C44),\\ &Z(L, 1, 3) = 0, \quad Z(L, 2, 0) = 0, \quad Z(L, 2, 1) = (-C11 + C33 + C22 - C44),\\ &Z(L, 2, 3) = 0, \quad Z(L, 3, 0) = I * (-C3 * C1 + C4 * C2 - C1 * C3 + C2 * C4),\\ &Z(L, 3, 1) = 0, \quad Z(L, 3, 2) = 0, \quad Z(L, 3, 3) = 0; \end{aligned}$$

$$\begin{aligned} X.\ X &= (X0.\ X0) - X1.\ X1 - X2.\ X2 - X3.\ X3,\\ XP.\ XP &= (XP0.\ XP0) - XP1.\ XP1 - XP2.\ XP2 - XP3.\ XP3; \end{aligned}$$

$$\begin{aligned} (X0.\ X0) = {}& C11 * C11 + C22 * C22 + C33 * C33 + C44 * C44\\ &+ 2 * (C11 * C22 - C11 * C33 - C11 * C33 - C11 * C44 - C22 * C33\\ &- C22 * C44 + C33 * C44); \end{aligned}$$

$X3.\ X3 = 0;$ $\qquad Y.\ Y = (Y0.\ Y0) - Y1.\ Y1 - Y2.\ Y2 - Y3.\ Y3;$

$XP3.\ XP3 = 0;$

$(Y0.\ Y0) = 0;$ $\qquad YP.\ YP = (YP0.\ YP0) - YP1.\ YP1 - YP2.\ YP2 - YP3.YP3;$

$(YP0.\ YP0) = 0;$

$$\begin{aligned} Y3.\ Y3 = {} & C11 * C11 + C22 * C22 + C33 * C33 + C44 * C44 \\ & + 2 * (-C11 * C22 - C11 * C33 + C11 * C44 + C22 * C33 \\ & - C22 * C44 - C33 * C44); \end{aligned}$$

$$\begin{aligned} YP3.\ YP3 = {} & C11P * C11P + C22P * C22P + C33P * C33P + C44P * C44P \\ & + 2 * (-C11P * C22P - \dots . \end{aligned}$$

This also necessitates the appropriate VECTOR declaration in the program. Namely, VECTOR X, Y, XP, YP, as well as INDEX declaration for the variables.

Appendix D

$$\begin{aligned}
\sigma_{11} &= -C_1^2[D_3 \sin\theta + (iD_2 - D_1)(\cos\theta \mp 1)] - C_3C_1D_0 \sin\theta, \\
\sigma_{12} &= C_1^2[D_3(\cos\theta \pm 1) - \sin\theta(iD_2 - D_1)] + C_3C_1D_0(\cos\theta \pm 1), \\
\sigma_{13} &= -C_1^2D_0 \sin\theta - C_3C_1[\sin\theta D_3 + (iD_2 - D_1)(\cos\theta \mp 1)], \\
\sigma_{14} &= C_1^2D_0(\cos \pm 1) + C_3C_1[D_3(\cos\theta \pm 1) - \sin\theta(iD_2 - D_1)], \\
\sigma_{21} &= -C_2^2[(iD_2 + D_1)\sin\theta + D_3(\cos\theta \mp 1)] + C_4C_2D_0(\cos\theta \mp 1), \\
\sigma_{22} &= C_2^2[(iD_2 + D_1)(\cos\theta \pm 1) - D_3 \sin\theta] + C_4C_2D_0 \sin\theta, \\
\sigma_{23} &= C_2^2D_0(\cos\theta \mp 1) - C_4C_2[\sin\theta(iD_2 + D_1) + D_3(\cos\theta \mp 1)], \\
\sigma_{24} &= C_2^2D_0 \sin\theta + C_4C_2[(\cos\theta \pm 1)(iD_2 + D_1) - D_3 \sin\theta], \\
\sigma_{31} &= -C_1C_3[D_3 \sin\theta + (iD_2 - D_1)(\cos\theta \mp 1) - C_3^2D_0 \sin\theta], \\
\sigma_{32} &= C_1C_3[D_3(\cos\theta \pm 1) - \sin\theta(iD_2 - D_1)] - C_3^2D_0(\cos\theta \pm 1), \\
\sigma_{33} &= -C_1C_3D_0 \sin\theta - C_3^2[\sin\theta D_3 + (iD_2 - D_1)(\cos\theta \mp 1)], \\
\sigma_{34} &= C_1C_3D_0(\cos\theta \pm 1) + C_3^2[D_3(\cos\theta \pm 1) - \sin\theta\,(iD_2 - D_1)], \\
\sigma_{41} &= -C_2C_4[(iD_2 + D_1)\sin\theta + D_3(\cos\theta \mp 1)] + C_4^2D_0(\cos\theta \mp 1), \\
\sigma_{42} &= C_2C_4[(iD_2 + D_1)(\cos\theta \pm 1) - D_3 \sin\theta] + C_4^2D_0 \sin\theta, \\
\sigma_{43} &= C_2C_4D_0(\cos\theta \mp 1) - C_4^2[\sin\theta(iD_2 + D_1) + D_3(\cos\theta \mp 1)], \\
\sigma_{44} &= C_2C_4D_0 \sin\theta + C_4^2[\cos\theta \pm 1)(iD_2 + D_1) - D_3 \sin\theta].
\end{aligned}$$

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